

# Inclusive decay of $B$ mesons into $D_s$ or $D_s^*$

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## Abstract

We compute the inclusive decay rates  $b \rightarrow D_s^-(D_s^{*-})c$  including lowest order QCD corrections on the quark legs, and compare with existing data. Unlike the short distance QCD corrections, that are of higher order, these corrections are of order  $\alpha_s$ . In the on-shell renormalization scheme and for  $\alpha_s(m_b) \cong 0.2$  we find a correction of  $-10\%$  to the inclusive rate computed using factorization. This gives a total rate  $BR(b \rightarrow D_s^-(D_s^{*-})c) \cong 8\%$  consistent within  $1\sigma$  with the measured value  $BR(B \rightarrow D_s^\pm X) = (10.0 \pm 2.5)\%$ . The general formulae given here include the case of vanishing mass for the final quark  $b \rightarrow D_s^-(D_s^{*-})u$ . The radiative correction to this rate is  $-17\%$ . We show in another place that this process can be useful for the measurement of the CKM matrix element  $V_{ub}$ . We also give the renormalized vertex at the interesting values  $q^2 = 0$  and  $q^2 = q_{max}^2$ , and compare with existing literature.

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# 1 Introduction

The inclusive rate of a  $\bar{B}$  meson decaying into a  $D_s^-$  meson is obtained using the spectator quark model shown in Fig. 1 :

$$\Gamma(\bar{B} \rightarrow D_s^- X) \cong \Gamma(b \rightarrow D_s^- c) + \Gamma(b \rightarrow D_s^{*-} c) \quad . \quad (1)$$

A main point in writing the approximation (1) is that, as we point out in ref. [1], the excited states  $D_s^{**}$  do not lead to  $D_s$  since their dominant decays are  $D_s^{**} \rightarrow D^{(*)}K$ . Moreover, the quark  $c$  dominates the inclusive rate, the quark  $u$  being CKM suppressed. Of course, there are also other mechanisms for producing the  $D_s$  of the right sign, namely through annihilation and exchange diagrams. However, as we discuss in detail in ref. [1], these mechanisms are suppressed. In this paper we will concentrate on the mechanism of Fig. 1, and compute the  $O(\alpha_s)$  radiative corrections to it.

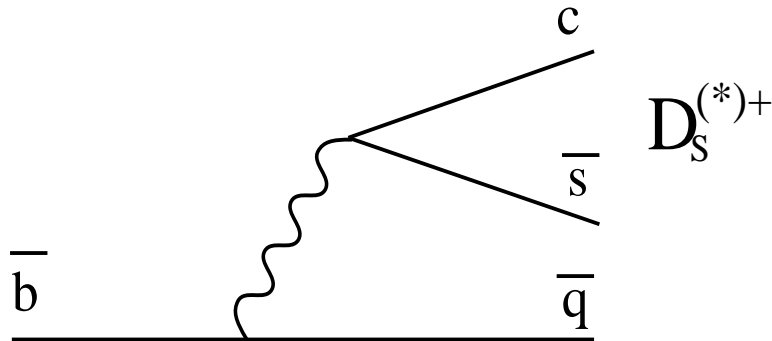


Figure 1: Spectator diagram for the decay  $\bar{b} \rightarrow D_s^{(*)+} \bar{q}$ .

From the effective weak Hamiltonian, using factorization and the spectator quark model, a straightforward calculation gives the rates

$$\Gamma^{(0)}(b \rightarrow D_s^- q) = \frac{G^2}{16\pi} |V_{qb}V_{cs}^*|^2 a_1^2 f_{D_s}^2 m_b^3 \sqrt{\lambda(1, r^2, \xi)} \left[ (1 - r^2)^2 - \xi(1 + r^2) \right] \quad (2)$$

$$\Gamma^{(0)}(b \rightarrow D_s^{*-} q) = \frac{G^2}{16\pi} |V_{qb}V_{cs}^*|^2 a_1^2 f_{D_s^*}^2 m_b^3 \sqrt{\lambda(1, r^2, \xi)} [(1 - r^2)^2 + \xi(1 + r^2 - 2\xi)] \quad (3)$$

where  $a_1$  is the QCD short distance factor

$$a_1 = C_2 + \frac{C_1}{N_c} = \frac{c_+ + c_-}{2} + \frac{c_+ - c_-}{2N_c} \quad . \quad (4)$$

Empirically, from exclusive decays, one finds  $a_1$  consistent with  $|a_1| \cong 1$ . For further convenience we have adopted the notation

$$r = \frac{m_q}{m_b} \quad \xi = \frac{q^2}{m_b^2} \quad (5)$$

where  $m_q$  is the final quark mass  $m_q = m_c$  or  $m_u$ ,  $q^2 = m_{D_s}^2$ , or  $m_{D_s^*}^2$ , and  $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$ .

To compute the order of magnitude of the expected branching fractions, let us tentatively use the following numerical values for the quark masses (see below for a detailed discussion of these parameters)

$$m_b = 5.0 \text{ GeV} \quad , \quad m_c = 1.6 \text{ GeV} \quad (6)$$

and  $m_u \cong 0$  and the decay constants [2]

$$f_{D_s} = 230 \text{ MeV} \quad f_{D_s^*} = 280 \text{ MeV} \quad . \quad (7)$$

Using  $\tau_B = 1.6 \text{ ps}$  and  $|V_{cb}| = 0.04$ , one obtains

$$BR^{(0)}(b \rightarrow D_s^- c) \cong 3.2 \% \quad BR^{(0)}(b \rightarrow D_s^{*-} c) \cong 6.8 \% \quad . \quad (8)$$

where the superindex (0) means using the factorization results (2), (3). Within the assumption (1), this yields

$$BR(\bar{B} \rightarrow D_s^- X) \cong BR^{(0)}(b \rightarrow D_s^- c) + BR^{(0)}(b \rightarrow D_s^{*-} c) \cong 10 \% \quad (9)$$

and, for completeness, with  $\frac{|V_{ub}|}{|V_{cb}|} = 0.08$ , let us give the  $b \rightarrow u$  branching ratios :

$$BR^{(0)}(b \rightarrow D_s^- u) \cong 2.6 \times 10^{-4} \quad BR^{(0)}(b \rightarrow D_s^{*-} u) \cong 4.9 \times 10^{-4} \quad . \quad (10)$$

The naive prediction (9) is consistent with the measured value [3]

$$BR(B \rightarrow D_s^\pm X) = (10.0 \pm 2.5) \% \quad . \quad (11)$$

The aim of this paper is to compute the lowest order QCD corrections to the process depicted in Fig. 1, i.e. to investigate how stable is the naive result (9) relatively to QCD radiative corrections. It is important to emphasize that the calculation of the radiative corrections will ask in particular for a detailed discussion of the quark masses, that we perform in Section 6.

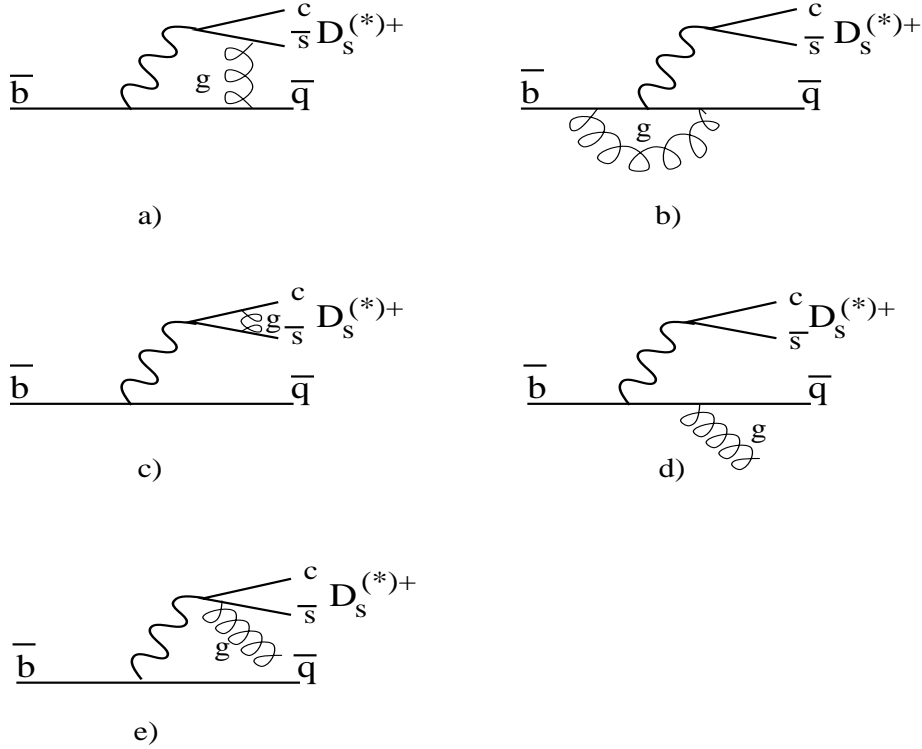


Figure 2: Diagrams leading to the production of  $D_s$  mesons.

Let us first notice that the leading logarithm corrections cancel at order  $\alpha_s$ . Indeed,

from the well known expressions at leading order  $c_{\pm}(\mu) = \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_W)} \right]^{d_{\pm}}$  where  $d_+ = -\frac{6}{23}$ ,  $d_- = \frac{12}{23}$ , one finds for the expression (4)  $a_1 = 1 + O(\alpha_s^2)$ . The correction is at most of second order, and this explains why the combination of short distance coefficients  $a_1$  is predicted by the theory to be close to one. This agrees with the measurements on class I exclusive decays [4], that yield an empirical  $|a_1| = 1.00 \pm 0.06$  [5] when one uses SVZ factorization [6] to calculate the matrix elements.

Figure 2 depicts the lower order QCD corrections. The contribution of Fig. 2a corresponds to gluons going from the lower to the upper vertex, they are UV divergent and contribute to the anomalous dimension. Figs. 2b,c are UV convergent if one adds the corresponding self-energy diagrams on the quark legs : this corresponds to the well-known fact that currents have no anomalous dimension. The fact that, as we have seen, the contribution of Fig. 2a vanishes at  $O(\alpha_s)$  follows from the color neutrality of the  $D_s^{(*)}$ . Indeed, the  $W$  produces a  $q\bar{q}$  in a color singlet. If the latter emits only one gluon which goes to the other vertex, and does not reabsorb any gluons, it becomes a color octet and cannot end into a  $D_s^{(*)}$ . Of course, this vanishing at  $O(\alpha_s)$  holds in general, not only for the leading  $\alpha_s \log \left( \frac{m_W^2}{\mu^2} \right)$  term in the combination  $a_1$  (4).

Since  $O(\alpha_s)$  leading log QCD corrections vanish, we will have to consider  $O(\alpha_s)$  corrections not containing the coefficient  $\log \left( \frac{m_W^2}{\mu^2} \right)$ , i.e. we will have to consider also the UV finite radiative corrections affecting *separately* the lower and upper quark legs. These corrections are IR divergent, and have to be combined with the Bremsstrahlung graphs on the same leg (Figs. 2d,e) to obtain the IR convergent physical result. At the upper vertex, however, all the gluon corrections of the type 2c, e (and the corresponding self-energies) have to be included in the vacuum to  $D_s^{(*)}$  matrix element represented by the decay constants  $f_{D_s}$  or  $f_{D_s^*}$ . Therefore, if we use a consistent

evaluation of the latter, we will not have to bother about gluons at the upper vertex.

In the following, we present an estimation of the  $O(\alpha_s)$  radiative corrections at the lower vertex ( $b$ - $q$  quark line), for arbitrary  $q$  quark mass and  $q^2$  carried by the current. Of course, the interesting cases are  $m_q = m_c$  or  $m_u$  and  $q^2 = m_{D_s}^2$  or  $m_{D_s^*}^2$ . Calculations of radiative corrections at the order aimed in this paper have been made in the past by Guberina, Peccei and Rückl [7], that computed the  $O(\alpha_s)$  corrections to the total rate  $b \rightarrow q_1 q_2 \bar{q}_3$  in the case of *vanishing masses* for the final quarks. The result is the sum of two contributions : the QCD translation of the Berman, Kinoshita and Sirlin calculation of QED corrections to  $\mu$  decay and the  $O(\alpha_s)$  correction to the ratio  $R$  in  $e^+e^-$  annihilation. Guberina et al. used, as we do in this paper, naive dimensional regularization (NDR) [8] for both UV and IR divergences. From their intermediate results for what is now for us the lower current in Fig. 2, we can compute, saturating with the tensors  $f_{D_s}^2 q^\mu q^\nu$  (for the  $D_s$ ) or  $f_{D_s^*}^2 (-g^{\mu\nu} q^2 + q^\mu q^\nu)$  (for the sum over the polarizations of the  $D_s^*$ ), with  $q^2 = m_{D_s}^2$  or  $m_{D_s^*}^2$ , the total rates  $\Gamma(b \rightarrow D_s^- u)$  and  $\Gamma(b \rightarrow D_s^* u)$ , since the  $u$  quark mass is approximately massless. We have done it, but we need to make the general calculation for a final massive quark  $q$ . In the limit  $m_q \rightarrow 0$  we recover the result that we have obtained from the intermediate stages worked out by Guberina et al. Here we will expose and discuss the general calculation.

Radiative corrections for weak processes taking into account the unequal quark masses have deserved much attention in the last years. In the nonleptonic three quark decay  $b \rightarrow q_1 q_2 \bar{q}_3$  with arbitrary quark masses, the calculation of the radiative corrections was performed by Hokim and Pham [9]. At lowest order, the current vertex had been computed by Gavela et al. and Halprin et al. [13], and by Paschalis

and Gounaris and by Schilcher et al. [12]. The radiative correction to the semileptonic heavy quark decays was performed by Cabibbo and Maiani and by Nir [16]. Recently, with the aim of applying the results to Heavy Quark Effective Theory, the vertex has been reconsidered by Voloshin and Shifman [14] and by Neubert [15].

The paper is planned as follows. Although these are, as the folklore says, “straight-forward but tedious calculations”, we will give some details, helped with Appendices in order to make the discussion simpler. In Section 2 we summarize the calculation of the renormalized vertex and in Appendix III we compute the limits at  $q^2 = 0$  and at  $q_{\text{max}}^2$  to compare with the existing literature. In Section 3 we give the corrected two-body rate. In Section 4 we summarize the calculation of the Bremsstrahlung rate. In Section 5 we discuss the analytical results and the necessary cancellations in  $\frac{1}{D-4}$  and of the mass singularities, and we deduce the  $m_q \rightarrow 0$  and  $q^2 \rightarrow 0$  limits of the final finite correction. In Section 6 we give numerical results for several interesting cases, along with a discussion of the values of the quark masses.

## 2 Renormalized vertex

We use NDR regularization and follow the Feynman rules and notations from the review article by Aoki et al. on the Standard Electroweak Model [10]. In particular, Dirac algebra is performed in  $D$  dimensions in Minkowski space and we work in the Feynman gauge.

A long but straightforward calculation gives, for the renormalized vertex

$$\Lambda_\mu^{(R)}(p_q, p_b) = \frac{16}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{6-D}{2}\right) \left(\frac{m_b}{\mu}\right)^{D-4} \left[ \gamma_\mu (1 - \gamma_5) A_L(r, \xi) + \frac{p_{b\mu}}{m_b} (1 + \gamma_5) B_+(r, \xi) + \frac{p_{q\mu}}{m_b} (1 + \gamma_5) C_+(r, \xi) \right]$$

$$+ \gamma_\mu (1 + \gamma_5) A_R(r, \xi) + \frac{p_{b\mu}}{m_b} (1 - \gamma_5) B_-(r, \xi) + \frac{p_{q\mu}}{m_b} (1 - \gamma_5) C_-(r, \xi) \Big] \quad (12)$$

where the coefficients of the different Dirac structures are :

$$\begin{aligned} A_L(r, \xi) &= \frac{1}{2} \left[ -\frac{D-3}{D-4} I_1 + \frac{(D-1)(1+r^{D-4})}{2(D-4)(D-3)} - \frac{1-\xi+r^2}{(D-4)(D-3)} I_2 \right. \\ &\quad \left. + \frac{r^2}{D-3} I_2 + \frac{1-r^2}{D-3} I_3 \right] \\ B_+(r, \xi) &= \frac{1}{2} I_4 \\ C_+(r, \xi) &= -\frac{1}{2} \left[ \frac{5-D}{D-3} I_3 + I_4 \right] \\ A_R(r, \xi) &= \frac{1}{2} r \frac{1}{D-3} I_2 \\ B_-(r, \xi) &= -\frac{1}{2} r \left[ \frac{2}{D-3} I_2 - \frac{D-1}{D-3} I_3 + I_4 \right] \\ C_-(r, \xi) &= \frac{1}{2} r (I_2 - 2I_3 + I_4) \quad . \end{aligned} \quad (13)$$

In the second term of the expression of  $A_L(r, \xi)$  we recognize the vertex counterterm, that we deduce from the Ward identity in Appendix I, and agrees with the result of [7]. The quantities  $I_i(\xi, r)$  are the integrals

$$\begin{aligned} I_1 &= \int_0^1 dx \left[ r^2(1-x) + x - \xi x(1-x) \right]^{D/2-2} \\ I_2 &= \int_0^1 dx \left[ r^2(1-x) + x - \xi x(1-x) \right]^{D/2-3} \\ I_3 &= \int_0^1 dx x \left[ r^2(1-x) + x - \xi x(1-x) \right]^{D/2-3} \\ I_4 &= \int_0^1 dx x^2 \left[ r^2(1-x) + x - \xi x(1-x) \right]^{D/2-3} \end{aligned} \quad (14)$$

whose expressions we give in Appendix II.

As a partial check of our calculation, we have computed the renormalized vertex at  $q^2 = 0$  and at  $q^2 = q_{max}^2$ . We give our results in Appendix III. The renormalized vertex for arbitrary masses and  $q^2$  had also been computed by Paschalis and Gounaris [12], using as infrared regulator a gluon mass  $\lambda$ , instead of dimensional regularization as



we use here. At  $q^2 = 0$  it had been computed by Gavela et al. [13] and by Hokim and Pham [9]. For the IR finite form factors we agree with their result. At  $q^2 = q_{max}^2$ , we agree with Paschalis and Gounaris [12], Voloshin and Shifman [14], and Neubert [15].

### 3 Two-body decay rate at order $\alpha_s$

After a rather long calculation, one finds, for the two-body decay rate at  $O(\alpha_s)$  :

$$\begin{aligned} \Gamma_{D_s}^{Two-body} = \Gamma_{D_s}^{(0)} & \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} g(D) \right. \\ & \left[ 2A_L + \frac{(1-r^2)^2 - \xi^2}{(1-r^2)^2 - \xi(1+r^2)} B_+ + \frac{(1-\xi-r^2)^2}{(1-r^2)^2 - \xi(1+r^2)} C_+ \right. \\ & + \frac{4r\xi}{(1-r^2)^2 - \xi(1+r^2)} A_R + \frac{r(1+\xi-r^2)^2}{(1-r^2)^2 - \xi(1+r^2)} B_- \\ & \left. \left. + \frac{r[(1-r^2)^2 - \xi^2]}{(1-r^2)^2 - \xi(1+r^2)} C_- \right] \right\} \end{aligned} \quad (15)$$

where  $A_L, \dots, C_-$  are understood to be functions of  $r$  and  $\xi$ , and the function  $g(D)$  is given by

$$g(D) = \frac{\Gamma\left(\frac{6-D}{2}\right)}{(2\pi)^{D-4} \text{ff}(4\pi)^{D-4}} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \left(\frac{\pi m_b^2}{\mu^2}\right)^{D-4} [\lambda(1, r^2, \xi)]^{(D-4)/2} . \quad (16)$$

Expanding in powers of  $D - 4$  one gets finally

$$\begin{aligned} \Gamma_{D_s}^{Two-body} = \Gamma_{D_s}^{(0)} & \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \frac{1}{(1-r^2)^2 - \xi(1+r^2)} \right. \\ & \left\{ 2(A_L + D_L) + \frac{(1-r^2)^2 - \xi^2}{(1-r^2)^2 - \xi(1+r^2)} B_+ + \frac{(1-\xi-r^2)^2}{(1-r^2)^2 - \xi(1+r^2)} C_+ \right. \\ & \left. + \frac{4r\xi}{(1-r^2)^2 - \xi(1+r^2)} A_R + \frac{r(1+\xi-r^2)^2}{(1-r^2)^2 - \xi(1+r^2)} B_- + \frac{r[(1-r^2)^2 - \xi^2]}{(1-r^2)^2 - \xi(1+r^2)} C_- \right\} \end{aligned} \quad (17)$$

where  $D_L(r, \xi)$  comes from the expansion of  $g(D)$  and the terms of order  $\frac{1}{D-4}$  in

$$A_L(r, \xi)$$

$$D_L = \left[ 1 - \frac{1}{2}(1 - \xi + r^2)I_2 \right] \left\{ -2 \log(2\pi) + \gamma - 1 + \log \left( \frac{\pi m_b^2}{\mu^2} \right) + \frac{1}{2} \log [\lambda(1, r^2, \xi)] \right\} . \quad (18)$$

In this expression  $I_2$  means the limit  $I_2^{D=4}$  that can be read from the formula in

Appendix II. Analogously, the other terms become

$$\begin{aligned} A_L &\cong \frac{1}{2} \left\{ \frac{1}{D-4} [3 - I_1 - (1 - \xi + r^2)I_2] - I_1 - 2 + \frac{3}{2} \log(r) + (1 - r^2)I_3 \right. \\ &\quad \left. + (1 - \xi + 2r^2)I_2 \right\} \\ B_+ &= \frac{1}{2} I_4 \\ C_+ &= -\frac{1}{2} (I_3 + I_4) \\ A_R &= \frac{1}{2} r I_2 \\ B_- &= -\frac{1}{2} r (2I_2 - 3I_3 + I_4) \\ C_- &= \frac{1}{2} r (I_2 - 2I_3 + I_4) \end{aligned} \quad (19)$$

where the limit  $D \rightarrow 4$  has been taken wherever it is possible, i.e. everywhere except in the first term of  $A_L$ .

## 4 Bremsstrahlung rate

From the expression of the real gluon emission amplitude

$$\begin{aligned} \mathcal{M}_a^{Brem}(\lambda) &= \frac{G}{\sqrt{2}} g_s \left[ \bar{u} \gamma_\mu (1 - \gamma_5) \frac{\not{p}_b - \not{k} + m_b}{m_b^2 - (p_b - k)^2} \not{\epsilon}^{(\lambda)} \frac{\lambda_a}{2} b + \right. \\ &\quad \left. \bar{u} \not{\epsilon}^{(\lambda)} \frac{\lambda_a}{2} \frac{\not{p}_u + \not{k} + m_u}{m_u^2 - (p_u + k)^2} \gamma_\mu (1 - \gamma_5) b \right] \bar{s} \gamma^\mu (1 - \gamma_5) c \end{aligned} \quad (20)$$

one gets the color and spin averaged rate for  $D_s$  emission :

$$|\overline{\mathcal{M}^{Brem}}|^2 = -\frac{1}{3} G^2 g_s^2 f_D^2 p_D^\mu p_D^\nu$$

$$\left[ \bar{u} \gamma_\mu (1 - \gamma_5) \frac{\not{p}_b - \not{k} + m_b}{m_b^2 - (p_b - k)^2} \gamma_\alpha b + \bar{u} \gamma_\alpha \frac{\not{p}_u + \not{k} + m_u}{m_u^2 - (p_u + k)^2} \gamma_\mu (1 - \gamma_5) b \right] \\ \left[ \bar{u} \gamma_\nu (1 - \gamma_5) \frac{\not{p}_b - \not{k} + m_b}{m_b^2 - (p_b - k)^2} \gamma^\alpha b + \bar{u} \gamma^\alpha \frac{\not{p}_u + \not{k} + m_u}{m_u^2 - (p_u + k)^2} \gamma_\nu (1 - \gamma_5) b \right]^+ . \quad (21)$$

Performing the Dirac algebra and putting quarks and gluons on-shell, one arrives at the expression for the rate :

$$\Gamma^{Brem} = -\frac{2}{3} \frac{1}{(2\pi)^{2D-3}} G^2 g_s^2 f_D^2 m_b^5 \\ \left\{ \left[ (1 - r^2)^2 - \xi(1 + r^2) \right] I^{-2,0} - \frac{1}{m_b^2} 2 \left[ (1 - r^2)^2 - \xi(1 + r^2) \right] I^{-1,0} \right. \\ \left. + \frac{1}{m_b^4} 4(1 + r^2) I^{0,0} + \frac{1}{m_b^2} 2 \left[ (1 - r^2)^2 - \xi(1 + r^2) \right] I^{0,-1} + r^2 \left[ (1 - r^2)^2 - \xi(1 + r^2) \right] I^{0,-2} \right. \\ \left. - \frac{1}{m_b^4} 2(1 + r^2) I^{-1,1} - \frac{1}{m_b^4} 2(1 + r^2) I^{1,-1} - (1 + r^2 - \xi) \left[ 1 - r^2 \right]^2 - \xi(1 + r^2) \right] I^{-1,-1} \Big\} \quad (22)$$

where  $I^{m,n}$  are the phase-space integrals in  $D$  dimensions :

$$I^{m,n} = \mu^{(m+n)(D-4)} \int \frac{d^{D-1}p_u}{2p_u^0} \frac{d^{D-1}p_D}{2p_D^0} \frac{d^{D-1}k}{2k^0} \delta^D(p_b - p_D - p_u - k) (p_b \cdot k)^m (p_u \cdot k)^n \quad (23)$$

that are functions of  $r$  and  $\xi$  and whose expressions are given in Appendix IV.

## 5 Discussion of the analytical results

### 5.1 $D_s$ rate

It is convenient to reorganize the expressions of the two-body and Bremsstrahlung decay rates in order to check the necessary cancellations, namely the  $\frac{1}{D-4}$  poles and the mass singularities. We can reorganize the expression for the two-body rate in the form :

$$\Gamma_{D_s}^{Two-body} = \Gamma_{D_s}^{(0)} \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ X + Y_L + \frac{(1 - r^2)^2 - \xi^2}{(1 - r^2)^2 - \xi(1 + r^2)} B_+ \right] \right\}$$

$$\begin{aligned}
& + \frac{(1 - \xi - r^2)^2}{(1 - r^2)^2 - \xi(1 + r^2)} C_+ + \frac{4\xi r}{(1 - r^2)^2 - \xi(1 + r^2)} A_R \\
& + \frac{r(1 + \xi - r^2)^2}{(1 - r^2)^2 - \xi(1 + r^2)} B_- + \frac{r[(1 - r^2)^2 - \xi^2]}{(1 - r^2)^2 - \xi(1 + r^2)} C_- \Big] \Big\} \quad (24)
\end{aligned}$$

where

$$\begin{aligned}
X = & \left[ 2 - \frac{1 - \xi + r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \right] \times \\
& \left[ \frac{1}{D - 4} - 2 \log(2\pi) + \gamma + \log \left( \frac{\pi m_b^2}{\mu^2} \right) \right] \quad (25)
\end{aligned}$$

and the new quantity  $Y_L$  is finite for  $D \rightarrow 4$ , given in Appendix V. Analogously, we can rewrite the Bremsstrahlung rate in the form :

$$\begin{aligned}
\Gamma_{D_s}^{Brem} = & - \Gamma_{D_s}^{(0)} \frac{4}{3} \frac{\alpha_s}{\pi} \left[ X + K^{-2,0} + r^2 K^{0,-2} - (1 + r^2 - \xi) K^{-1,-1} \right. \\
& - 2K^{-1,0} + 2K^{0,-1} + \frac{4(1 + r^2)}{(1 - r^2)^2 - \xi(1 + r^2)} K^{0,0} \\
& \left. - \frac{2(1 + r^2)}{(1 - r^2)^2 - \xi(1 + r^2)} K^{-1,1} - \frac{2(1 + r^2)}{(1 - r^2)^2 - \xi(1 + r^2)} K^{1,-1} \right] \quad (26)
\end{aligned}$$

where  $X$  is the quantity defined by (25) and the expressions  $K^{m,n}$  are given in Appendix V. We observe that  $X$ , that contains the  $\frac{1}{D-4}$  terms and the terms in  $\log\left(\frac{m_b}{\mu}\right)$ , cancels between the two-body and the Bremsstrahlung rates, as expected. To check the cancellation of the rest of the mass singularities, we need to perform an expansion in powers of  $m_q$  or in powers of  $r$ . From the expression (24) and Appendix II we obtain, for  $r \rightarrow 0$  :

$$\Gamma_{D_s}^{Two-body} = \Gamma_{D_s}^{(0)} \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ [D(r, \xi)]_{r \rightarrow 0} + [F_V(r, \xi)]_{r \rightarrow 0} \right] \right\} \quad (27)$$

$[D(r, \xi)]_{r \rightarrow 0}$  contains the  $\frac{1}{D-4}$  and the divergent terms as  $m_q \rightarrow 0$  :

$$\begin{aligned}
[D(r, \xi)]_{r \rightarrow 0} = & \left[ 2 - 2 \log \left( \frac{1 - \xi}{r} \right) \right] \left[ \frac{1}{D - 4} - 2 \log(2\pi) + \gamma + \log \left( \frac{\pi m_b^2}{\mu^2} \right) \right] \\
& + \log(r) \left[ -\frac{5}{2} + 2 \log(1 - \xi) \right] + \log^2(r) \quad (28)
\end{aligned}$$

and  $[F_V(r, \xi)]_{r \rightarrow 0}$  is the surviving finite piece :

$$\begin{aligned} [F_V(r, \xi)]_{r \rightarrow 0} &= -3 - \frac{\pi^2}{6} - 3 \log^2(1 - \xi) + 4 \log(1 - \xi) + \frac{1}{\xi} \log(1 - \xi) \\ &+ Sp(1 - \xi) + \log \xi \log(1 - \xi) \quad . \end{aligned} \quad (29)$$

The Spence function  $Sp(z)$  is defined in Appendix II. Analogously, we obtain :

$$\Gamma_{D_s}^{Brem} = -\Gamma_{D_s}^{(0)} \frac{4}{3} \frac{\alpha_s}{\pi} \{ [D(r, \xi)]_{r \rightarrow 0} + [F_B(r, \xi)]_{r \rightarrow 0} \} \quad (30)$$

$$\begin{aligned} [F_B(r, \xi)]_{r \rightarrow 0} &= -\frac{21}{4} + \frac{\pi^2}{3} + \frac{13}{2} \log(1 - \xi) + \log(\xi) \log(1 - \xi) + Sp(\xi) \\ &- 3 \log^2(1 - \xi) + \frac{\xi}{1 - \xi} \log(\xi) \quad . \end{aligned} \quad (31)$$

We observe also that the singular terms in  $\log(r)$  and  $\log^2(r)$  contained in  $[D(r, \xi)]_{r \rightarrow 0}$  cancel among the two-body and Bremsstrahlung rates.

Then, it follows, for  $m_q \rightarrow 0$ , the total rate :

$$\begin{aligned} \Gamma_{D_s}^{Two-body} + \Gamma_{D_s}^{Brem} &= \Gamma_{D_s}^{(0)} \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ \frac{9}{4} - \frac{\pi^2}{3} - 2Sp(\xi) - \log(\xi) \log(1 - \xi) \right. \right. \\ &\left. \left. + \frac{1}{\xi} \log(1 - \xi) - \frac{5}{2} \log(1 - \xi) - \frac{\xi}{1 - \xi} \log(\xi) \right] \right\} \end{aligned} \quad (32)$$

that gives, at  $q^2 = 0$ , i.e.  $\xi = 0$  :

$$\Gamma_{D_s}^{Two-body} + \Gamma_{D_s}^{Brem} = \Gamma_{D_s}^{(0)} \left[ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left( \frac{5}{4} - \frac{\pi^2}{3} \right) \right] \quad . \quad (33)$$

## 5.2 $D_s^*$ rate

The calculation of the  $D_s^*$  rate follows along the same lines, with the replacement

$$f_D^2 p_D^\mu p_D^\nu \rightarrow f_{D^*}^2 m_{D^*}^2 \sum_{\lambda} \varepsilon^{(\lambda)\mu} \varepsilon^{*(\lambda)\nu} = f_{D^*}^2 p_{D^*}^\mu p_{D^*}^\nu - f_{D^*}^2 m_{D^*}^2 g^{\mu\nu} \quad . \quad (34)$$

The difference between the  $D_s^*$  and  $D_s$  rates is thus a term proportional to  $m_{D^*}^2$ , i.e.  $q^2$  or  $\xi$ . In the limit  $\xi \rightarrow 0$  one must recover the same decay rate. We obtain, for the two-body rate :

$$\begin{aligned} \Gamma_{D_s^*}^{Two-body} = \Gamma_{D_s^*}^{(0)} & \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ X + Y_L + \frac{\lambda(1, r^2, \xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} B_+ \right. \right. \\ & + \frac{\lambda(1, r^2, \xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} C_+ - \frac{12r\xi}{(1-r^2)^2 + \xi(1+r^2-2\xi)} A_R \\ & \left. \left. + \frac{r\lambda(1, r^2, \xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} B_- + \frac{r\lambda(1, r^2, \xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} C_- \right] \right\} \quad (35) \end{aligned}$$

and for the Bremsstrahlung :

$$\begin{aligned} \Gamma_{D_s^*}^{Brem} = -\Gamma_{D_s^*}^{(0)} & \frac{4}{3} \frac{\alpha_s}{\pi} \left[ X + K^{-2,0} + r^2 K^{0,-2} - (1+r^2-\xi) K^{-1,-1} - 2K^{-1,0} + \right. \\ & + 2K^{0,-1} + \frac{4(1+r^2)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} K^{0,0} - \frac{2(1+r^2+2\xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} K^{-1,1} \\ & \left. - \frac{2(1+r^2+2\xi)}{(1-r^2)^2 + \xi(1+r^2-2\xi)} K^{1,-1} \right] . \quad (36) \end{aligned}$$

Again, as expected,  $X$ , that contains the  $\frac{1}{D-4}$  terms, cancels between the two-body and the Bremsstrahlung rates. As for the mass singularities, we find in the case of the  $D_s^*$  exactly the same expression for  $[D]_{r \rightarrow 0}$  that cancels among the two-body and the Bremsstrahlung rates. The finite result is, in this case :

$$\begin{aligned} \Gamma_{D_s^*}^{Two-body} + \Gamma_{D_s^*}^{Brem} = \Gamma_{D_s^*}^{(0)} & \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ 2 - \frac{\pi^2}{3} - 2Sp(\xi) - \log(\xi) \log(1-\xi) \right. \right. \\ & \left. \left. - \frac{5+4\xi}{2(1+2\xi)} \log(1-\xi) - \frac{3-\xi-10\xi^2}{4(1-\xi)(1+2\xi)} - \frac{\xi(1-\xi-2\xi^2)}{(1-\xi)^2(1+2\xi)} \log(\xi) \right] \right\} \quad (37) \end{aligned}$$

that gives, at  $\xi = 0$  :

$$\Gamma_{D_s^*}^{Two-body} + \Gamma_{D_s^*}^{Brem} = \Gamma_{D_s^*}^{(0)} \left[ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left( \frac{5}{4} - \frac{\pi^2}{3} \right) \right] \quad (38)$$

i.e., the same correction than for the  $D_s$ , as expected.

## 6 Numerical results

To give the numerical results, let us parametrize the rate, including the QCD corrections, under the form :

$$\Gamma(b \rightarrow D_s^- q) = \Gamma^{(0)}(b \rightarrow D_s^- q) \left[ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \eta(\xi_{D_s}, r) \right] \quad (39)$$

and analogously for the  $D_s^*$

$$\Gamma(b \rightarrow D_s^{*-} q) = \Gamma^{(0)}(b \rightarrow D_s^{*-} q) \left[ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \eta^*(\xi_{D_s^*}, r) \right] \quad (40)$$

The functions  $\eta(\xi, r)$  and  $\eta^*(\xi, r)$  can be read respectively from the finite sums of the two-body and Bremsstrahlung corrections, formulae (24), (26) and (35), (36). The particular values  $\eta(\xi_{D_s}, r)$  and  $\eta^*(\xi_{D_s^*}, r)$  correspond respectively to both functions for  $q^2 = m_{D_s}^2$  and  $m_{D_s^*}^2$ . At fixed  $r$  these are slowly varying functions of  $\xi$ . For  $r = 0$ , these functions are given by the relatively simple expressions (32) and (37), and for  $r = r_c = \frac{m_c}{m_b}$  we use the full expressions. The functions *decrease* monotonously with  $\xi$ , the dependence on  $q^2$  is mild, and the values at the interesting values of low  $q^2$  (i.e.  $\xi \lesssim 0.2$ ) do not differ significantly from the  $q^2 = 0$  value. For  $\xi \rightarrow \xi_{max}$  there is a logarithmic singularity, of the form  $\log(1 - \xi)$  for  $r = 0$ , smoothed out by the phase space  $\Gamma^{(0)}(\xi_{max}) = 0$ . The functions  $\eta(\xi, 0)$ ,  $\eta(\xi, r_c)$ ,  $\eta^*(\xi, 0)$  and  $\eta^*(\xi, r_c)$  decrease by about 50 % between  $\xi = 0$  and  $\xi = 0.6 \xi_{max}$ . The correction depends weakly on the ratio  $r = \frac{m_c}{m_b}$  in the expected range of quark masses. The radiative corrections depend on the ratios  $\xi_D$ ,  $\xi_{D^*}$  and  $r$  and on  $\alpha_s(\mu)$ . We take  $\mu = m_b$  and we adopt the value  $\alpha_s(m_b) = 0.2$ .

For the quark masses, we take pole masses from a fit to the semileptonic decay rate  $b \rightarrow c \ell^- \bar{\nu}_\ell$  with QCD corrections at one loop [16], to be consistent with the same order

that we compute here. The semileptonic decay rate reads, in this approximation :

$$\Gamma(b \rightarrow c \ell^- \bar{\nu}_\ell) = \frac{G^2 m_b^5}{192 \pi^3} |V_{cb}|^2 f_{PS}(r) \left[ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} f_{RC}(r) \right] \quad (41)$$

where the phase space  $f_{PS}(r)$  and radiative correction  $f_{RC}(r)$  functions depend on  $r = \frac{m_c}{m_b}$  and are given in [16]. Setting  $r = 0.3$ , we obtain, from the semileptonic branching ratio 11 %,

$$m_b = 4.85 \text{ GeV} \quad m_c = 1.45 \text{ GeV} \quad . \quad (42)$$

The mass difference  $m_b - m_c = 3.40 \text{ GeV}$  compares well with the value  $m_b - m_c = (3.43 \pm 0.04) \text{ GeV}$  obtained in the  $1/m$  expansion of the Heavy Quark Effective Theory [17]. Therefore the pole masses that we choose (42), at one loop, seem reasonable.

With these values, and  $m_u \cong 0$ , the results are the following, for the  $u$ -quark :

$$\begin{aligned} \frac{4}{3} \frac{\alpha_s}{\pi} \eta(\xi_{D_S}, 0) &= -0.168 \\ \frac{4}{3} \frac{\alpha_s}{\pi} \eta^*(\xi_{D_S^*}, 0) &= -0.159 \end{aligned} \quad (43)$$

and for the  $c$ -quark :

$$\begin{aligned} \frac{4}{3} \frac{\alpha_s}{\pi} \eta(\xi_{D_S}, r_c) &= -0.096 \\ \frac{4}{3} \frac{\alpha_s}{\pi} \eta^*(\xi_{D_S^*}, r_c) &= -0.108 \quad . \end{aligned} \quad (44)$$

Then, using these values, the corrected branching ratios will be, from (8) and (10) :

$$\begin{aligned} BR(b \rightarrow D_s^- c) &\cong 2.6 \% \\ BR(b \rightarrow D_s^{*-} c) &\cong 5.4 \% \quad . \end{aligned} \quad (45)$$

Our conclusion is that the sum including radiative corrections

$$BR(b \rightarrow D_s^- c) + BR(b \rightarrow D_s^{*-} c) \cong 8 \% \quad (46)$$



is still in agreement within  $1\sigma$  with the measurement (11). Also, the moderate radiative correction to the processes  $b \rightarrow D_s^{(*)-} u$  reinforces our argument about using the spectrum  $\bar{B} \rightarrow D_s^- X_u$  in the measurement of the CKM matrix element  $V_{ub}$  [1].

A number of remarks is in order here. First, from the relation at one loop between the pole mass and the running  $\overline{MS}$  mass

$$m = \bar{m}(\bar{m}) \left[ 1 + \frac{4}{3} \frac{\alpha_s(\bar{m})}{\pi} \right] \quad (47)$$

at first order in  $\alpha_s$ , from (41) one obtains :

$$\Gamma(b \rightarrow c \ell \bar{\nu}_\ell) = \frac{G^2 [\bar{m}_b(\bar{m}_b)]^5}{192\pi^3} |V_{cb}|^2 f_{PS}(r) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(\bar{m}_b)}{\pi} [5 + f_{RC}(r)] \right\} \quad (48)$$

and in formulas (39) and (40) when  $[\bar{m}_b(\bar{m}_b)]^3$  is substituted to  $m_b^3$ , a term  $\frac{4}{3} \frac{\alpha_s}{\pi} \times 3$  has to be added. Taking  $\alpha_s(\bar{m}_b) = 0.22$  we get from (47)

$$\bar{m}_b(\bar{m}_b) = 4.43 \text{ GeV} \quad . \quad (49)$$

The values of the pole mass (42) and of the  $\overline{MS}$  running mass (49) are respectively smaller and larger than the values recently quoted in the literature from the analysis of semileptonic  $b$  decay, the reason being that a partial resummation of higher order diagrams is made that enhances the radiative corrections by roughly a factor 2 (see for example [17]). The interest of considering the  $\overline{MS}$  mass is that the series is Borel summable, while using the pole mass, there is a renormalon ambiguity, cancelled by another renormalon ambiguity in the pole mass. Ball et al. [17] quote, as central values,  $m_b = 5.05 \text{ GeV}$ ,  $m_c = 1.62 \text{ GeV}$ , and  $\bar{m}_b(\bar{m}_b) = 4.23 \text{ GeV}$ ,  $\bar{m}_c(\bar{m}_c) = 1.29 \text{ GeV}$ ,

leading to consistent results for the semileptonic rate in both the  $\overline{MS}$  and on-shell schemes.

Concerning the radiative corrections at higher orders in the processes  $b \rightarrow D_s^{(*)-} c$ , one part of the radiative corrections has the same topology than in semileptonic decays (simply at a given value of  $q^2$  instead of integrating over  $q^2$ ), and we could expect that these radiative corrections would be similarly enhanced by higher orders. This is by the way what happens at order  $\alpha_s$  : the correction that we obtain for  $b \rightarrow D_s^{(*)-} c$  is very close to the one obtained at the same order in the semileptonic case [16]. This would imply, e.g. in the on-shell renormalization scheme, a larger radiative correction by a factor 2 [17], but consistently also a larger  $m_b = 5.05$  GeV, leading grosso modo to the same results (45) and (46). We must keep in mind however that, already at second order in  $\alpha_s$ , we could have another type of corrections, absent in the semileptonic decay, that break factorization (e.g. two or more gluons linking the  $D_s$  to the quark legs). We can only hope that these corrections will be small, like it is the case for the short distance QCD coefficient (4), that empirically is very close to 1.

Work remains to be done, in particular the calculation of the QCD-corrected spectrum  $b \rightarrow D_s^- c$  to be compared with the measured spectrum [18] of  $B \rightarrow D_s^\pm X$ . It would be interesting to check in particular if the same function describing the  $b$ -quark Fermi motion [19] fits the semileptonic spectrum and the inclusive  $D_s$  one as well.

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## Appendix I. On-shell renormalization, Ward identity and vertex counterterm

The bare self-energy of a quark of mass  $m$  writes

$$\Sigma(p) = A(p^2)\not{p} + B(p^2)$$

with

$$\begin{aligned} A(p^2) &= \frac{4}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{4-D}{2}\right) m^{D-4} \frac{4}{D} F\left(\frac{4-D}{2}, 2; \frac{D}{2} + 1; \xi\right) \\ B(p^2) &= -\frac{4}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{4-D}{2}\right) m^{D-4} \frac{2D}{D-2} m F\left(\frac{4-D}{2}, 1; \frac{D}{2}; \xi\right). \end{aligned}$$

To proceed with the on-shell renormalization it is convenient to expand  $\Sigma(p)$  in powers of  $\not{p} - m$ , that gives :

$$\Sigma(p) = -\frac{4}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{4-D}{2}\right) m^{D-4} \left[ \frac{2(D+1)}{D-3} m + \frac{1-D}{D-3} \not{p} \right] + \dots$$

The unknowns  $a, b$  in the self-mass counterterm

$$\sigma(p) = a\not{p} + b$$

will be fixed by the on-shell renormalization conditions [10] :

$$\bar{u}(p) \Sigma^{(R)}(p) \Big|_{\not{p}=m} = 0 \quad \Sigma^{(R)}(p) u(p) \Big|_{\not{p}=m} = 0$$

where  $u(p)$  is a solution of the Dirac equation with mass  $m$  :  $(\not{p} - m) u(p) = 0$ , and  $\Sigma^{(R)}(p) = \Sigma(p) + \sigma(p)$ . These conditions yield

$$\sigma(p) = \frac{4}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{4-D}{2}\right) m^{D-4} \left[ \frac{2(D+1)}{D-3} m + \frac{1-D}{D-3} \not{p} \right] + \dots$$

On the other hand, as we use NDR, that preserves chiral symmetry, both the self-energy  $\sigma(p)$  and current  $\lambda_\mu^{qb}$  counterterms will satisfy separately the Ward identity

(an overall factor  $\frac{g}{2\sqrt{2}}V_{qb}$  is understood, where  $g$  is the weak coupling) [10] [11] :

$$(p_q - p_b)^\mu \lambda_\mu^{qb} = \left[ \sigma^q(p_q) T^{(+)} (1 - \gamma_5) - (1 + \gamma_5) T^{(+)} \sigma^b(p_b) \right] - i M_W \lambda^{qb}$$

where  $\lambda^{qb}$  is the counterterm of the coupling of the unphysical Higgs and  $T^{(+)}b = q$ .

One gets immediately

$$\lambda_\mu^{ub} = \frac{4}{3} g_s^2 \frac{1}{2^D \pi^{D/2}} \Gamma\left(\frac{6-D}{2}\right) \frac{D-1}{(D-3)(D-4)} \left[ m_q^{D-4} + m_b^{D-4} \right] \gamma_\mu (1 - \gamma_5) \quad .$$

## Appendix II. Vertex integrals

An expansion is made of the vertex integrals  $I_i(\xi, r)$  up to first power of  $D - 4$  in the cases in which this is necessary ( $I_1$  and  $I_2$ ) :

$$I_1 = 1 - \frac{D-4}{2} \left[ 2 + \frac{1-r^2-\xi}{2\xi} \log(r^2) + \frac{\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{1+r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}} \right) \right]$$

$$\begin{aligned} I_2 = & \frac{1}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{1+r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}} \right) \left[ 1 + \frac{1}{2}(D-4) \log \xi \right] \\ & + \frac{1}{2}(D-4) \frac{1}{\sqrt{\lambda(1, r^2, \xi)}} \left[ -2Sp \left( \frac{2\sqrt{\lambda(1, r^2, \xi)}}{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}} \right) + 2Sp \left( \frac{2\sqrt{\lambda(1, r^2, \xi)}}{1-r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}} \right) \right. \\ & - \frac{1}{2} \log^2 \left( \frac{1-r^2+\xi-\sqrt{\lambda(1, r^2, \xi)}}{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}} \right) + \frac{1}{2} \log^2 \left( \frac{1-r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}}{1-r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}} \right) \\ & + \log^2 \left( \frac{1-r^2+\xi-\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) - \log^2 \left( \frac{1-r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) \\ & \left. - \log^2 \left( \frac{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) + \log^2 \left( \frac{1-r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) \right] \end{aligned}$$

$$I_3 = -\frac{1}{2\xi} \log(r^2) - \frac{1-r^2-\xi}{2\xi\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{1+r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}} \right)$$

$$I_4 = \frac{1}{\xi} + \frac{1 - r^2 - \xi}{2\xi^2} \log(r^2) + \frac{(1 - r^2)^2 + \xi(\xi - 2)}{2\xi^2 \sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) .$$

The Spence function  $Sp(z)$  is defined by

$$Sp(z) = \int_0^1 dt \frac{\log(t)}{t - \frac{1}{z}}$$

$$Sp(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad , \quad |z| \leq 1 \quad .$$

In the limit  $q^2 \rightarrow q_{max}^2$ , since  $\xi_{max} \rightarrow (1 - r)^2$  and  $\lambda(1, r^2, \xi_{max}) \rightarrow 0$ , one obtains :

$$I_1(\xi_{max}, r) = 1 - (D - 4) \left[ 1 + \frac{r}{1 - r} \log(r) \right]$$

$$I_2(\xi_{max}, r) = \frac{1}{r} \left[ 1 + (D - 4) \left( 1 + \frac{1}{1 - r} \log(r) \right) \right]$$

$$I_3(\xi_{max}, r) = -\frac{1}{1 - r} \left[ \frac{1}{1 - r} \log(r) + 1 \right]$$

$$I_4(\xi_{max}, r) = \frac{1 + r}{(1 - r)^2} + \frac{2r}{(1 - r)^3} \log(r) \quad .$$

Another limit used in the text is  $q^2$  or  $\xi \rightarrow 0$  :

$$I_1(0, r) = 1 - \frac{D - 4}{2} \left[ 1 + \frac{1}{1 - r^2} \log(r^2) - \log(r^2) \right]$$

$$I_2(0, r) = -\frac{1}{1 - r^2} \log(r^2) - \frac{1}{4} (D - 4) \frac{1}{1 - r^2} \log^2(r^2)$$

$$I_3(0, r) = \frac{1}{1 - r^2} + \frac{r^2}{(1 - r^2)^2} \log(r^2)$$

$$I_4(0, r) = -\frac{r^4}{(1 - r^2)^3} \log(r^2) + \frac{1 - 3r^2}{2(1 - r^2)^2} \quad .$$

### Appendix III. Renormalized vertex at $q^2 = 0$ and $q^2 = q_{max}^2$

**Vertex at  $q^2 = 0$**

Defining the quark couplings by the expansion of the renormalized vertex ( $q = p_b - p_u = p - p'$ ) :

$$\Lambda_{\mu}^{(R)}(p', p) = g_V \gamma_{\mu} - g_A \gamma_{\mu} \gamma_5 + \frac{g_M}{2m_b} i \sigma_{\mu\nu} q^{\nu} + \frac{g_T}{2m_b} i \sigma_{\mu\nu} q^{\nu} \gamma_5 + \frac{g_S}{2m_b} q_{\mu} + \frac{g_P}{2m_b} q_{\mu} \gamma_5$$

we find, at  $q^2 = 0$  :

$$\begin{aligned}
g_V &= 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left\{ -\frac{5}{8} + \frac{1}{16} \frac{1+r^2}{1-r^2} \log^2(r^2) + \frac{-3-7r^2-8r+2(1+r)^2}{16(1-r^2)} \log(r^2) \right. \\
&\quad \left. + \frac{1}{2} \left[ \frac{1}{D-4} + \log\left(\frac{\pi m_b^2}{\mu^2}\right) + \gamma - 2 \log(2\pi) \right] \left[ 2 + \frac{1+r^2}{1-r^2} \log(r^2) \right] \right\} \\
g_A &= 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left\{ -\frac{5}{8} + \frac{1}{16} \frac{1+r^2}{1-r^2} \log^2(r^2) + \frac{-3-7r^2+8r+2(1-r)^2}{16(1-r^2)} \log(r^2) \right. \\
&\quad \left. + \frac{1}{2} \left[ \frac{1}{D-4} + \log\left(\frac{\pi m_b^2}{\mu^2}\right) + \gamma - 2 \log(2\pi) \right] \left[ 2 + \frac{1+r^2}{1-r^2} \log(r^2) \right] \right\} \\
g_M &= \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -\frac{1-r}{2(1-r^2)} + \frac{r(1-r)}{2(1-r^2)^2} \log(r^2) \right] \\
g_T &= \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -\frac{1+r}{2(1-r^2)} - \frac{r(1+r)}{2(1-r^2)^2} \log(r^2) \right] \\
g_S &= \frac{4}{3} \frac{\alpha_s}{\pi} \left[ \frac{1}{(1-r^2)^2} (1+2r-2r^2-r^3) + \frac{r}{2(1-r^2)^3} (3+r-r^2-3r^3) \log(r^2) \right] \\
g_P &= \frac{4}{3} \frac{\alpha_s}{\pi} \left[ \frac{1}{(1-r^2)^2} (1-2r-2r^2+r^3) - \frac{r}{2(1-r^2)^3} (3-r-r^2+3r^3) \log(r^2) \right] .
\end{aligned}$$

The finite couplings  $g_M$ ,  $g_T$ ,  $g_S$ ,  $g_P$  agree with former calculations by Halprin et al., and by Gavela et al. [13], and by Hokim and Pham [9]. We find infrared divergent results for  $g_V$ ,  $g_A$ , as we found in [13] with a gluon mass  $\lambda$  as infrared cut-off. In our expressions for  $g_V$  and  $g_A$  the last term, that vanishes for  $r \rightarrow 1$ , cancels when the two-body decay rate is added to the Bremsstrahlung rate, just as it happened with the  $\lambda$  regulator. In the equal mass limit  $r \rightarrow 1$ , we find, keeping fixed the infrared

cut-off, the expected result :

$$\Lambda_\mu^{(R)}(p', p) = \gamma_\mu (1 - \gamma_5) + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ \frac{1}{2} \gamma_\mu \gamma_5 - \frac{1}{2} \frac{1}{2m} i\sigma_{\mu\nu} (p^\nu - p'^\nu) + \frac{7}{6} \frac{1}{2m} (p_\mu - p'_\mu) \gamma_5 \right].$$

**Vertex at  $q^2 = q_{max}^2$**

From (13) and the limit of the integrals (14) at  $q^2 \rightarrow q_{max}^2$  given in Appendix II, we obtain

$$\begin{aligned} \Lambda_\mu^{(R)}(p', p) = & \gamma_\mu \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -1 - \frac{1+r}{2(1-r)} \log(r) \right] \right\} \\ & - \gamma_\mu \gamma_5 \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -\frac{3}{2} - \frac{1+r}{2(1-r)} \log(r) \right] \right\} \\ & + \frac{4}{3} \frac{\alpha_s}{\pi} \left\{ \frac{1}{2m_b} i\sigma_{\mu\nu} q^\nu \left[ \frac{1}{2(1-r)} \log(r) \right] + \frac{1}{2m_b} i\sigma_{\mu\nu} q^\nu \gamma_5 \left[ \frac{1}{1-r} + \frac{1+r}{2(1-r)^2} \log(r) \right] \right. \\ & \left. + \frac{q_\mu}{2m_b} \left[ -\frac{1}{1-r} - \frac{1+r}{2(1-r)^2} \log(r) \right] + \frac{q_\mu}{2m_b} \gamma_5 \left[ \frac{2(1+r)}{(1-r)^2} - \frac{1+r^2-10r}{2(1-r)^3} \log(r) \right] \right\} \end{aligned}$$

in agreement with Paschalis and Gounaris [12]. In terms of four-velocities, this reads,

$$\begin{aligned} \Lambda_\mu^{(R)}(p, p') = & \gamma_\mu \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -1 - \frac{3}{4} \frac{1+r}{1-r} \log(r) \right] \right\} \\ & - \gamma_\mu \gamma_5 \left\{ 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left[ -2 - \frac{3}{4} \frac{1+r}{1-r} \log(r) \right] \right\} \\ & + \frac{4}{3} \frac{\alpha_s}{\pi} \left\{ v_\mu \left[ -\frac{1}{2(1-r)} - \frac{r}{2(1-r)^2} \log(r) \right] + v'_\mu \left[ \frac{r}{2(1-r)} + \frac{r}{2(1-r)^2} \log(r) \right] \right. \\ & \left. + v_\mu \gamma_5 \left[ \frac{3+r}{2(1-r)^2} + \frac{r(5-r)}{2(1-r)^3} \log(r) \right] + v'_\mu \gamma_5 \left[ -\frac{r(1+3r)}{2(1-r)^2} - \frac{r(5r-1)}{2(1-r)^3} \log(r) \right] \right\} \end{aligned}$$

that is also in agreement with the calculation of Neubert (first reference [15]). Finally, using the Gordon identities for unequal masses

$$\begin{aligned} \bar{u}(p') \gamma_\mu u(p) &= \frac{1}{2} \bar{u}(p') \left[ v_\mu + v'_\mu - i\sigma_{\mu\nu} (v^\nu - v'^\nu) \right] u(p) \\ \bar{u}(p') \left[ v_\mu + v'_\mu - i\sigma_{\mu\nu} (v^\nu - v'^\nu) \right] \gamma_5 u(p) &= 0 \end{aligned}$$

one obtains the form

$$\begin{aligned}
\Lambda_\mu^{(R)}(p', p) = & \gamma_\mu (1 - \gamma_5) \\
& + \frac{4}{3} \frac{\alpha_s}{\pi} \left\{ \gamma_\mu \left[ -\frac{3}{2} - \frac{3}{4} \frac{1+r}{1-r} \log(r) \right] - \gamma_\mu \gamma_5 \left[ -2 - \frac{3}{4} \frac{1+r}{1-r} \log(r) \right] \right. \\
& - \frac{1}{2} \frac{1}{2} i \sigma_{\mu\nu} (v^\nu - v'^\nu) + \frac{1}{2} i \sigma_{\mu\nu} (v^\nu - v'^\nu) \gamma_5 \left[ \frac{3(1+r)}{2(1-r)} + \frac{3r}{(1-r)^2} \log(r) \right] \\
& + \frac{1}{2} (v_\mu - v'_\mu) \left[ -\frac{1+r}{2(1-r)} - \frac{2r}{2(1-r)^2} \log(r) \right] \\
& \left. + \frac{1}{2} (v_\mu - v'_\mu) \gamma_5 \left[ \frac{3r^2 + 2r + 3}{2(1-r)^2} + \frac{4r(1+r)}{2(1-r)^3} \log(r) \right] \right\} \quad .
\end{aligned}$$

The vector and axial vector couplings have been written down by Neubert [15], and we agree with his result. In the equal mass limit  $r \rightarrow 1$  we recover, as we must, the result of equal masses at  $q^2 = 0$  given above.

## Appendix IV. Phase space integrals

Two-body phase space

$$\begin{aligned}
I_0 = & \mu^{4-D} \int \frac{d^{D-1} p_u}{2p_u^0} \frac{d^{D-1} p_D}{2p_D^0} \delta^D(p_b - p_D - p_u) \\
I_0 = & \frac{\pi^{(D-1)/2}}{2^{D-2}} \frac{1}{\Gamma\left(\frac{D-1}{2}\right)} \left(\frac{m_b}{\mu}\right)^{D-4} \left[\lambda(1, r^2, \xi)\right]^{(D-3)/2} \quad .
\end{aligned}$$

Three-body phase space integrals

$$I^{m,n} = \mu^{(m+n)(D-4)} \int \frac{d^{D-1} p_u}{2p_u^0} \frac{d^{D-1} p_D}{2p_D^0} \frac{d^{D-1} k}{2k^0} \delta^D(p_b - p_D - p_u - k) (p_b \cdot k)^m (p_u \cdot k)^n$$

$$I^{-2,0} = \pi^2 \frac{1}{m_b^2} \left\{ \frac{1}{D-4} \sqrt{\lambda(1, r^2, \xi)} \right.$$



$$\begin{aligned}
& + \left[ \log \left( \frac{\pi m_b^2}{\mu^2} \right) + \gamma - 2 + 3 \log \left( \sqrt{\lambda(1, r^2, \xi)} \right) - \log(r) - \log(\xi^{1/2}) \right] \sqrt{\lambda(1, r^2, \xi)} \\
& - \left( 1 - \xi + r^2 \right) \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - \left( 1 + \xi - r^2 \right) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \Big\}
\end{aligned}$$

$$\begin{aligned}
I^{0,-2} &= \pi^2 \frac{1}{m_b^2} \frac{1}{r^2} \left\{ \frac{1}{D-4} \sqrt{\lambda(1, r^2, \xi)} \right. \\
& + \left[ \log \left( \frac{\pi m_b^2}{\mu^2} \right) + \gamma - 2 + 3 \log \left( \sqrt{\lambda(1, r^2, \xi)} \right) - \log(r) - \log(\xi^{1/2}) \right] \sqrt{\lambda(1, r^2, \xi)} \\
& \left. - 2(1 - \xi) \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - (1 - \xi - r^2) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
I^{-1,-1} &= \pi^2 \frac{1}{m_b^2} \left\{ \frac{2}{D-4} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \right. \\
& + \left[ 2 \log \left( \frac{\pi m_b^2}{\mu^2} \right) + 2\gamma - 2 + 4 \log(r) + \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) \right. \\
& + 3 \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \left. \right] \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \\
& + \log(r) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) - \frac{\pi^2}{3} + 3Sp \left[ \left( \frac{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}}{2r} \right)^2 \right] \\
& \left. - Sp \left[ \frac{1 - r^2 + \xi - \sqrt{\lambda(1, r^2, \xi)}}{2} \right] - Sp \left[ \frac{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}}{2} \right] \right\}
\end{aligned}$$

$$I^{-1,0} = \pi^2 \left\{ \frac{1}{2} \sqrt{\lambda(1, r^2, \xi)} - r^2 \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \right\}$$

$$-\xi \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \Bigg\}$$

$$I^{0,-1} = \pi^2 \left\{ -\frac{1}{2} \sqrt{\lambda(1, r^2, \xi)} + (1 - \xi) \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \right. \\ \left. - \xi \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\}$$

$$I^{-1,1} = \pi^2 m_b^2 \frac{1}{4} \left\{ \frac{1}{4} (1 - 3r^2 + 5\xi) \sqrt{\lambda(1, r^2, \xi)} \right. \\ \left. + r^4 \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - \xi (2 - 2r^2 + \xi) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\}$$

$$I^{1,-1} = \pi^2 m_b^2 \frac{1}{8} \left\{ -\frac{1}{2} (3 - r^2 - 5\xi) \sqrt{\lambda(1, r^2, \xi)} \right. \\ \left. + 2 \left[ (1 - \xi)^2 + 2r^2 \xi \right] \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \right. \\ \left. - 2\xi (2 - 2r^2 - \xi) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\}$$

$$I^{0,0} = \pi^2 m_b^2 \frac{1}{4} \left\{ \frac{1}{2} (1 + r^2 + \xi) \sqrt{\lambda(1, r^2, \xi)} \right. \\ \left. - 2r^2 (1 - \xi) \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - 2\xi (1 - r^2) \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\} .$$

## Appendix V

Here we define necessary expressions that enter into formulae (24), (26) and (35), (36), that are not given in the course of the text. For the two-body decay rate we have the new expression :

$$\begin{aligned}
Y_L = & \left[ 2 - \frac{1 - \xi + r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \right] \left\{ -1 + \frac{1}{2} \log [\lambda(1, r^2, \xi)] \right\} \\
& - 2 + \frac{1 - r^2 - \xi}{4\xi} \log(r^2) + \frac{\sqrt{\lambda(1, r^2, \xi)}}{4\xi} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \\
& + \frac{3}{2} \log(r) - \frac{1 - \xi + r^2}{\sqrt{\lambda(1, r^2, \xi)}} \left\{ \frac{1}{2} \log \xi \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \right. \\
& - Sp \left( \frac{2\sqrt{\lambda(1, r^2, \xi)}}{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}} \right) + Sp \left( \frac{2\sqrt{\lambda(1, r^2, \xi)}}{1 - r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}} \right) \\
& - \frac{1}{4} \log^2 \left( \frac{1 - r^2 + \xi - \sqrt{\lambda(1, r^2, \xi)}}{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}} \right) + \frac{1}{4} \log^2 \left( \frac{1 - r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}}{1 - r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}} \right) \\
& + \frac{1}{2} \log^2 \left( \frac{1 - r^2 + \xi - \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) - \frac{1}{2} \log^2 \left( \frac{1 - r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) \\
& \left. - \frac{1}{2} \log^2 \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) + \frac{1}{2} \log^2 \left( \frac{1 - r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) \right\} \\
& + \frac{1 - \xi + 2r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \\
& - \frac{1 - r^2}{2\xi} \left[ \log(r^2) + \frac{1 - r^2 - \xi}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{1 + r^2 - \xi - \sqrt{\lambda(1, r^2, \xi)}} \right) \right] .
\end{aligned}$$

And for the Bremsstrahlung :

$$\begin{aligned}
K^{-2,0} = & \left[ -2 + 3 \log \left( \sqrt{\lambda(1, r^2, \xi)} \right) - \log(r) - \log(\xi^{1/2}) \right] \\
& - \frac{1 - \xi + r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 + r^2 - \xi + \sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - \frac{1 + \xi - r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1 - r^2 + \xi + \sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right)
\end{aligned}$$

$$K^{0,-2} = \frac{1}{r^2} \left\{ \left[ -2 + 3 \log \left( \sqrt{\lambda(1, r^2, \xi)} \right) - \log(r) - \log(\xi^{1/2}) \right] \right. \\ \left. - \frac{2(1-\xi)}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{2r} \right) - \frac{1-\xi-r^2}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}}{2\xi^{1/2}} \right) \right\}$$

$$K^{-1,-1} = \frac{1}{\sqrt{\lambda(1, r^2, \xi)}} \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \times \\ \left[ -2 + 4 \log(r) + \log \left( \frac{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) + 3 \log \left( \frac{1+r^2-\xi+\sqrt{\lambda(1, r^2, \xi)}}{2r} \right) \right] \\ + \frac{1}{\sqrt{\lambda(1, r^2, \xi)}} \left[ \log(r) \log \left( \frac{1-r^2+\xi+\sqrt{\lambda(1, r^2, \xi)}}{2\xi} \right) - \frac{\pi^2}{3} \right. \\ \left. + 3 \operatorname{Sp} \left[ \left( \frac{1+r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}}{2r} \right)^2 \right] \right. \\ \left. - \operatorname{Sp} \left[ \frac{1-r^2+\xi-\sqrt{\lambda(1, r^2, \xi)}}{2} \right] - \operatorname{Sp} \left[ \frac{1+r^2-\xi-\sqrt{\lambda(1, r^2, \xi)}}{2} \right] \right]$$

$$K^{m,n} = \frac{1}{\pi^2 \sqrt{\lambda(1, r^2, \xi)}} I^{m,n} \quad \text{for } (m, n) = (-1, 0), (0, -1)$$

$$K^{m,n} = \frac{1}{\pi^2 m_b^2 \sqrt{\lambda(1, r^2, \xi)}} I^{m,n} \quad \text{for } (m, n) = (-1, 1), (1, -1)$$

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